

10/29

Triple Integrals

IDEA: Integrate functions of 3 variables

Remark: all the hard work to up the dimension was already done

• 1-variable \rightarrow 2-variable was the hardest part

Conceptually, this is no different from double integrals, but pictures are hard.

Main Technique:

$\iiint_R f(x, y, z) dV$ is computable via an iterated integral

- Same as before, order of integration is more-or-less up to us as long as we parameterize appropriately

Ex: compute $\iiint_E (xy + z^2) dV$ for $E = [0, 2] \times [0, 1] \times [0, 3]$
(Rectangular prism)

$$= \left\{ \begin{array}{l} 0 \leq x \leq 2 \\ (x, y, z): 0 \leq y \leq 1 \\ 0 \leq z \leq 3 \end{array} \right\}$$

Sol:
$$\int_{x=0}^2 \int_{y=0}^1 \int_{z=0}^3 (xy + z^2) dz dy dx$$

Innermost:
$$\int_{z=0}^3 (xy + z^2) dz = \left[xyz + \frac{1}{3} z^3 \right]_{z=0}^3 = (3xy + 9) - 0$$

Middle:
$$\int_{y=0}^1 (3xy + 9) dy = \left[\frac{3}{2} xy^2 + 9y \right]_{y=0}^1 = \left(\frac{3}{2} x + 9 \right) - 0$$

Outermost:
$$\int_{x=0}^2 \left(\frac{3}{2} x + 9 \right) dx = \left[\frac{3}{4} x^2 + 9x \right]_{x=0}^2 = \left(\frac{3}{4} \cdot 4 + 18 \right) - 0 = \boxed{21}$$

Ex. Compute $\iiint_R (2x-y) dV$ where $R = \{(x,y,z): 0 \leq z \leq 2, 0 \leq y \leq z^2, 0 \leq x \leq y-z\}$

Note: this parameterization has the form:

$$\{(x,y,z): C_1 \leq z \leq C_2, g_1(z) \leq y \leq g_2(z), h_1(y,z) \leq x \leq h_2(y,z)\}$$

This has the same form as what we liked when we computed double integrals

Sol.
$$= \int_{z=0}^2 \int_{y=0}^{z^2} \int_{x=0}^{y-z} 2(x-y) dx dy dz$$

Innermost:
$$\int_{x=0}^{y-z} (2x-y) dx = \left[x^2 - xy \right]_{x=0}^{y-z} = (y-z)^2 - (y-z)y - 0$$

$$= y^2 - 2yz + z^2 - y^2 + yz = z^2 - yz$$

Middle:
$$\int_{y=0}^{z^2} (z^2 - yz) dy = \left[yz^2 - \frac{1}{2} y^2 z \right]_{y=0}^{z^2} = z^2 \cdot z^2 - \frac{1}{2} (z^2)^2 z - 0 = z^4 - \frac{1}{2} z^5$$

Outermost:
$$\int_{z=0}^2 (z^4 - \frac{1}{2} z^5) dz = \left[\frac{1}{5} z^5 - \frac{1}{12} z^6 \right]_{z=0}^2 = \frac{32}{5} - \frac{64}{12} - 0 = \frac{32}{5} - \frac{16}{3}$$

$$= \boxed{\frac{16}{15}}$$

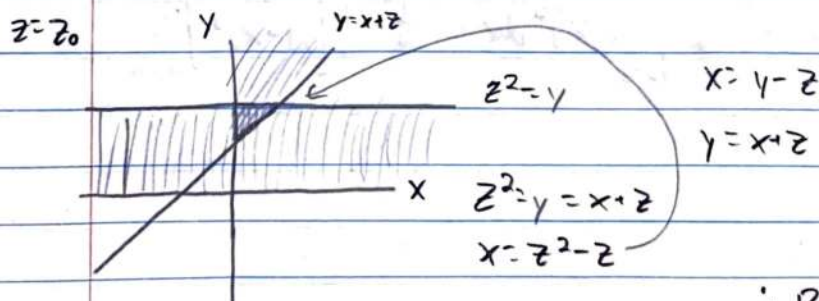
Remark on Reparameterization:

To change the order of integration, we must reparameterize to look something like this:

For this region R in the previous example, to change the order to $dy dx dz$, we need a reparameterization of the form

$$R = \{(x,y,z): C_1 \leq z \leq C_2, g_1(z) \leq x \leq g_2(z), h_1(x,z) \leq y \leq h_2(x,z)\}$$

Look at $z = z_0$ cross-section - effectively fixes z as a constant



$$0 \leq x \leq z^2 - z$$

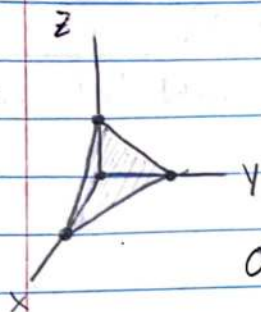
$$x - z \leq y \leq z^2$$

$$\therefore R = \left\{ (x,y,z): \begin{array}{l} 0 \leq z \leq 2 \\ 0 \leq x \leq z^2 - z \\ x - z \leq y \leq z^2 \end{array} \right.$$

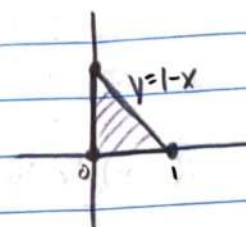
Note: will be fully worked in a PDF on the website

Ex: Compute the volume of the tetrahedron T with vertices $(0,0,0)$ $(1,0,0)$ $(0,1,0)$ $(0,0,1)$

Picture:



XY-Shadow



$$0 \leq x \leq 1$$

$$0 \leq y \leq 1-x$$

$$0 \leq z \leq 1-x-y$$

$$\vec{u} = \langle 1-0, 0-0, 0-1 \rangle = \langle 1, 0, -1 \rangle$$

$$\vec{v} = \langle 0-0, 1-0, 0-1 \rangle = \langle 0, 1, -1 \rangle$$

$$\vec{n} = \vec{u} \times \vec{v} = \langle 1, 1, 1 \rangle$$

\therefore the plane has formula $\vec{n} \cdot (\vec{x} - \vec{p}) = 0$

$$\text{i.e. } \langle 1, 1, 1 \rangle \cdot \langle x-0, y-0, z-1 \rangle = x+y+z-1=0$$

$$z = 1-x-y$$

$$\therefore T = \{(x,y,z) : 0 \leq x \leq 1, 0 \leq y \leq 1-x, 0 \leq z \leq 1-x-y\}$$

$$\therefore \text{Vol}(T) = \iiint 1 \, dV = \int_{x=0}^1 \int_{y=0}^{1-x} \int_{z=0}^{1-x-y} 1 \, dz \, dy \, dx$$

$$= \int_{x=0}^1 \int_{y=0}^{1-x} [z]_{z=0}^{1-x-y} \, dy \, dx = \int_{x=0}^1 \int_{y=0}^{1-x} (1-x-y-0) \, dy \, dx$$

$$\int_{x=0}^1 \left[y - xy - \frac{1}{2}y^2 \right]_{y=0}^{1-x} \, dx = \int_{x=0}^1 \left((1-x) - x(1-x) - \frac{1}{2}(1-x)^2 \right) \, dx$$

$$= \int_{x=0}^1 \left((1-x)^2 - \frac{1}{2}(1-x)^2 \right) \, dx = \frac{1}{2} \int_{x=0}^1 (1-x)^2 \, dx = \frac{1}{2} - \frac{1}{3} \left[(1-x)^3 \right]_{x=0}^1$$

$$= -\frac{1}{6}(0-1) = \boxed{\frac{1}{6}}$$